

## 201(4): Relativistic General Relativity. II

### Spherical Spacetime

The method of general relativity is a spherical spacetime all rest or  $\Phi$  infinitesimal line element:

$$ds^2 = c^2 d\tau^2 = m(r, t) c^2 dt^2 - n(r, t) dr^2 - r^2 d\theta^2 \quad - (1)$$

in  $\Phi$  plane:

$$dz^2 = 0. \quad - (2)$$

By definition:

$$c^2 d\tau^2 = m(r, t) c^2 dt^2 - \underline{dr} \cdot \underline{dr} \quad - (3)$$

where:

$$\underline{dr} \cdot \underline{dr} = v^2 dt^2 = n(r, t) dr^2 + r^2 d\theta^2 \quad - (4)$$

$$s.o. \quad v^2 = n(r, t) \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2 \quad - (5)$$

The standard method uses  $\Phi$  Lagrangian:

$$\begin{aligned} \mathcal{L} = \frac{1}{2} mc^2 &= \frac{1}{2} m m(r, t) c^2 \left( \frac{dt}{d\tau} \right)^2 - \frac{1}{2} m n(r, t) \left( \frac{dr}{d\tau} \right)^2 \\ &\quad - \frac{1}{2} m r^2 \left( \frac{d\theta}{d\tau} \right)^2. \quad - (6) \end{aligned}$$

The Euler Lagrange equation gives the total energy  $E$ , and the total angular momentum  $L$ :

$$E = m(r, t) m c^2 \frac{dt}{d\tau} \quad - (7)$$

$$L = m r^2 \frac{d\theta}{d\tau} \quad - (8)$$

where

$$\frac{dt}{d\tau} = \left( m(r) - \frac{v^2}{c^2} \right)^{-1/2} \quad - (9)$$

$$\frac{d\theta}{d\tau} = \frac{d\theta}{dt} \frac{dt}{d\tau} = \omega \frac{dt}{d\tau} \quad - (10)$$

Therefore:

$$\omega = \frac{d\theta}{dt} = \frac{c b n(r)}{r^2} \quad - (11)$$

where

$$b = \frac{L c}{E} \quad - (12)$$

From eq. (1):

$$m c^2 = m n(r, t) c^2 \left( \frac{dt}{d\tau} \right)^2 - m n(r, t) \left( \frac{dr}{d\tau} \right)^2 - m r^2 \left( \frac{d\theta}{d\tau} \right)^2$$

$$= \frac{E^2}{m(r, t) m c^2} - m n(r, t) \left( \frac{dr}{d\tau} \right)^2 - \frac{L^2}{m r^2} \quad - (13)$$

Therefore:

$$m n(r, t) \left( \frac{dr}{d\tau} \right)^2 = \frac{E^2}{m(r, t) m c^2} - m c^2 - \frac{L^2}{m r^2} \quad - (14)$$

In this equation:

$$\frac{dr}{d\tau} = \frac{dr}{d\theta} \frac{d\theta}{d\tau} = \frac{L}{m r^2} \frac{dr}{d\theta} \quad - (15)$$

So:

$$\frac{m n(r, t) L^2}{m^2 r^4} \left( \frac{dr}{d\theta} \right)^2 = \frac{E^2}{m(r, t) m c^2} - m c^2 - \frac{L^2}{m r^2} \quad - (16)$$

$$\left( \frac{dr}{d\theta} \right)^2 = \frac{m r^4}{n(r, t) L^2} \left( \frac{E^2}{m(r, t) m c^2} - m c^2 - \frac{L^2}{m r^2} \right) \quad - (17)$$

$$= \frac{r^4}{n(r, t) m(r, t)} \frac{E^2}{L^2 c^2} - \frac{m^2 c^2 r^4}{n(r, t) L^2} - \frac{r^2}{n(r, t)}$$

$$= \frac{r^4}{n(r, t)} \left( \frac{1}{n(r, t) b^2} - \left( \frac{1}{a^2} + \frac{1}{r^2} \right) \right)$$

where  $a = \frac{L}{mc}$ ,  $b = \frac{Lc}{E}$  — (19)

The orbital equation is: — (20)

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{n(r,t)} \left( \frac{1}{m(r,t)b^2} - \left( \frac{1}{a^2} + \frac{1}{r^2} \right) \right)$$

Now use:

$$\begin{aligned} \left(\frac{dr}{dt}\right)^2 &= \omega^2 \left(\frac{dr}{d\theta}\right)^2 = \frac{c^2 b^2 m^2(r,t)}{r^4} \left(\frac{dr}{d\theta}\right)^2 \quad - (21) \\ &= \frac{c^2 b^2 m^2(r,t)}{n(r,t)} \left[ \frac{1}{m(r,t)b^2} - \left( \frac{1}{a^2} + \frac{1}{r^2} \right) \right] \\ &= c^2 \left[ \frac{m(r,t)}{n(r,t)} - b^2 \frac{m^2(r,t)}{n(r,t)} \left( \frac{1}{a^2} + \frac{1}{r^2} \right) \right]. \quad - (22) \end{aligned}$$

Therefore:

$$\left(\frac{dr}{dt}\right)^2 = c^2 \frac{m(r,t)}{n(r,t)} \left( 1 - b^2 m(r,t) \left( \frac{1}{a^2} + \frac{1}{r^2} \right) \right) \quad - (23)$$

From eq. (5):

$$\begin{aligned} v^2 &= c^2 m(r,t) \left( 1 - b^2 m(r,t) \left( \frac{1}{a^2} + \frac{1}{r^2} \right) \right) \\ &\quad + \left( \frac{c b m(r)}{r} \right)^2 \quad - (24) \end{aligned}$$

t) This result is not given in HFT 194 and 196b.

Therefore:

$$v^2 = c^2 m(r,t) \left( 1 - \left( \frac{mc^2}{E} \right)^2 m(r,t) \right) \quad - (25)$$

From eqs. (7), (8) and (9):

$$m(r,t) = \frac{1}{2} \left( \frac{E}{mc^2} \right) \left( 1 \pm \left( 1 - 4 \frac{v^2}{c^2} \left( \frac{mc^2}{E} \right)^2 \right)^{1/2} \right)$$

Eliminate  $v^2$  between eqs. (25) and (26) to

give:

$$m(r,t) - \frac{1}{2} \left( \frac{E}{mc^2} \right) = \pm \frac{1}{2} \left( \frac{E}{mc^2} \right) \left( 1 - 4 \frac{v^2}{c^2} \left( \frac{mc^2}{E} \right)^2 \right)^{1/2} \quad - (27)$$

Square both sides:

$$\left( m(r,t) - \frac{1}{2} \left( \frac{E}{mc^2} \right) \right)^2 = \frac{1}{4} \left( \frac{E}{mc^2} \right)^2 \left( 1 - 4 \frac{v^2}{c^2} \left( \frac{mc^2}{E} \right)^2 \right)$$

$$\begin{aligned} &= \frac{1}{4} \left( \frac{E}{mc^2} \right)^2 - \frac{v^2}{c^2} \\ &= \frac{1}{4} \left( \frac{E}{mc^2} \right)^2 - m(r) \left( 1 - \left( \frac{mc^2}{E} \right)^2 m(r) \right) \\ &= m^2(r,t) - \left( \frac{E}{mc^2} \right) m(r,t) + \frac{1}{4} \left( \frac{E}{mc^2} \right)^2 \end{aligned} \quad - (28)$$

Therefore:

$$5) \quad n^2(r, t) \left( 1 - \left( \frac{mc^2}{E} \right)^2 \right) = m(r, t) \left( \frac{E}{mc^2} - 1 \right) \quad - (29)$$

i.e

$$m(r, t) = \frac{\frac{E}{mc^2} - 1}{1 - \left( \frac{mc^2}{E} \right)^2} \quad - (30)$$

$$= \left( \frac{E - mc^2}{mc^2} \right) \left( \frac{E^2}{E^2 - (mc^2)^2} \right)$$

Finally use:

$$E^2 - (mc^2)^2 = (E + mc^2)(E - mc^2) \quad - (31)$$

So

$$m(r, t) = \frac{E}{mc^2(E + mc^2)}$$

$$\boxed{m(r, t) = \left( \frac{E}{mc^2} \right)^2 \left( 1 + \frac{E}{mc^2} \right)^{-1}} \quad - (32)$$

This is a constant of motion for all spherical spacetimes. The following theorems are referred.

1) All those that depend on:

$$m(r) = n^{-1}(r) = 1 - \frac{r_0}{r} \quad - (33)$$

6) These theories include perihelion precession, light bending by gravitation, black holes, gravitational red shift, black hole theory, frame dragging theory and so on.

2) The big bang theory which depends on the Robertson Walker metric:

$$ds^2 = c^2 dt^2 - \frac{a^2(t)}{1 - kr^2} dr^2 - a^2(t) r^2 d\theta^2 \quad - (34)$$

As is well known this is a particular case of eq. (1) with:

$$n(r, t) = 1. \quad - (35)$$

From eq. (32) it is seen immediately that:

$$\left( \frac{E}{mc^2} \right)^2 = 1 + \frac{E}{mc^2} \quad - (36)$$

This is a quadratic of type:

$$x^2 - x - 1 = 0, \quad - (37)$$

$$\text{So: } x = \frac{E}{mc^2} = \frac{1}{2} \left( 1 \pm (1 + 4) \right) \quad - (38)$$

$$= \frac{1}{2a} \left( -b \pm (b^2 - 4ac)^{1/2} \right) \quad - (39)$$

$$\therefore \frac{E}{mc^2} = \frac{1}{2} (1 \pm \sqrt{5})$$

which is absurd because  $E$  is constant.