

208(10) : Statement of the Complete Orbital
problem.

Computer algebra is needed to solve the
problem for any orbital function $f(\theta)$. The problem
is to solve:

$$\frac{d\omega}{d\theta} = -F\omega \quad - (1)$$

where

$$F = \frac{d^2 f / d\theta^2}{df / d\theta} \quad - (2)$$

and

$$f = \left(\frac{r}{f_1} \right)^2 \quad - (3)$$

$$f_1 = \frac{dr}{d\theta} \quad - (4)$$

Eq. (1) can be written as:

$$\frac{d\omega}{\omega} = -F d\theta \quad - (5)$$

so

$$\log_e \omega = - \int F d\theta \quad - (6)$$

$$\text{Let } \int F d\theta = F_1(\theta) + C, \quad - (7)$$

where C is the constant of integration.

2) So:

$$\omega = \omega_0 \exp \left(- \int F d\theta \right) - (8)$$

$$\omega = \omega_0 \exp \left(- \left(F_1(\theta) + C \right) \right) - (9)$$

The constant angular frequency ω_0 is needed for correct units, and C is the constant of integration.

For any orbital function (4) the computer algebra gives the orbital angular frequency ω as a function of θ , ω_0 and C . If ω can be observed in astronomy as a function of θ then $F_1(\theta)$ can be evaluated by computer. For ω_0 and C can be found numerically.

This is a non-trivial problem in general.

Solar System

The main feature of the observed orbit is the precessing ellipse:

$$r = \frac{d}{1 + e \cos(x\theta)} - (10)$$

3) but the orbit may be considerably more complicated than this, with small modulations. The main angular frequency is found from eq. (10), which gives:

$$\frac{dr}{d\theta} = \left(\frac{ex}{d} \right) r^2 \sin(x\theta), \quad - (11)$$

$$= \left(\frac{ex}{d} \right) \frac{d^2 \sin(x\theta)}{(1 + e \cos(x\theta))^2} \quad - (12)$$

$$\therefore \left(\frac{dr}{d\theta} \right)^2 = \left(exd \right)^2 \frac{\sin^2(x\theta)}{(1 + e \cos(x\theta))^4} \quad - (13)$$

and

$$f = \left(\frac{r}{exd} \right)^2 \left(\frac{(1 + e \cos(x\theta))^4}{\sin^2(x\theta)} \right)$$

$$f = \left(\frac{1}{ex} \right)^2 \left(\frac{1 + e \cos(x\theta)}{\sin(x\theta)} \right)^2 \quad - (14)$$

The angular frequency of the orbit may be found from eqs. (9) and (14), and expressed in terms of θ , ω_0 and C .

4) To an excellent approximation, the orbit of a planet in the solar system is given by:

$$e \rightarrow 1, \quad x \rightarrow 1. \quad - (15)$$

In this case the angular frequency is Newtonian and given by:

$$\omega_N = \frac{L}{m} \frac{1}{r^2} \quad - (16)$$

where

$$r = \frac{d}{1 + e \cos \theta} \quad - (17)$$

Here L is the constant angular momentum of the system, and m is the mass of the planet. So:

$$\omega_N = \left(\frac{L}{md^2} \right) (1 + e \cos \theta)^2 \quad - (18)$$

Now assume that the general solution (9) can be expressed as:

$$\omega = \omega_N + A\omega_1 \quad - (19)$$

where A is a constant. More generally:

$$\omega = \omega_N + A\omega_1 + B\omega_2 + C\omega_3 + \dots \quad - (20)$$

3) Hence:

$$\omega = \omega_0 \exp \left(- \left(F_1(\theta) + c \right) \right) \quad - (21)$$
$$= \omega_N + A\omega_1 + B\omega_2 + \dots$$

Conclusion

The equation of motion of the new relativity, eq. (1), shows that the Newtonian angular frequency ω_N is perturbed.

Iterative Procedure

- 1) First assume that the orbit is a precessing ellipse, so F_1 is found from eq. (14). Proceed to find ω in terms of ω_0 , F_1 and C . Find ω_N from eq. (18). ~~Find~~ Assume that:

$$\omega = \omega_N + A\omega_1 \quad - (22)$$

and find A and ω_1 .

- 2) Assume that the orbit is a modulated or perturbed precessing ellipse, i.e. a precessing ellipse superimposed on which

b) are very small oscillations of various angular frequencies. In this model the processing eq. (10) is represented by a Fourier Series:

$$r = \frac{a_0}{2} + \sum_{y=1}^{\infty} (a_y \cos(y\theta) + b_y \sin(y\theta)) \quad - (22)$$

It is well known that any function can be synthesized from a Fourier series, and that this is a well developed subject.

Therefore find the Fourier series needed for a processing ellipse, and perturb it, using numerical methods. Again find A and ω_1 from eq. (22). Determine how much the perturbation has changed A and ω_1 . There are many methods such as this possible.

Pure Newtonian Approximation

In this case:

$$\frac{L}{md^3} (1 + \epsilon \cos \theta)^2 = \omega_0^2 \exp(- (F_1(\theta) + c)) \quad - (23)$$

$$7) \quad \epsilon \ll 1 \quad - (24)$$

$$\text{So:} \quad \exp\left(-\left(F_1(\theta) + C\right)\right) \sim \frac{L}{n d^3 \omega_0} \quad - (25)$$

$$\therefore e \quad F_1(\theta) + C \sim -\log_e\left(\frac{L}{n d^3 \omega_0}\right) \quad - (26)$$

$$\text{and} \quad F_1(\theta) \sim \text{constant} \quad - (27)$$

For small ϵ :

$$r = \frac{d}{1 + \epsilon \cos \theta} \sim d(1 - \epsilon \cos \theta) \quad - (28)$$

$$\frac{dr}{d\theta} \sim d \epsilon \sin \theta \quad - (29)$$

$$f = \left(r \frac{d\theta}{dr}\right)^2 \sim \frac{d}{\epsilon \sin \theta} \quad - (30)$$

$$\frac{df}{d\theta} \sim -\frac{d}{\epsilon} \frac{\cos \theta}{\sin^2 \theta} \quad - (31)$$

$$\frac{d^2 f}{d\theta^2} \sim \frac{d}{\epsilon} \left(\frac{\sin^2 \theta - 2 \cos^2 \theta}{\sin^3 \theta} \right) \quad - (32)$$

$$F \sim \frac{2 \cos^2 \theta - \sin^2 \theta}{\cos \theta \sin \theta} \quad - (33)$$

and:

8)

$$F_1 + C = -\log_e \left(\frac{L}{md^2 \omega_0} \right)$$

$$= \int \frac{2\cos^2 \theta - \sin^2 \theta}{\cos \theta \sin \theta} d\theta \quad - (34)$$

So:

$$\int \frac{2\cos^2 \theta - \sin^2 \theta}{\cos \theta \sin \theta} d\theta = f_1(\theta) + f_1(0)$$

- (35)

where $f_1(0) \gg f_1(\theta) \quad - (36)$

i.e. the constant of integration $f_1(0)$ is chosen to be:

$$f_1(0) = -\log_e \left(\frac{L}{md^2 \omega_0} \right) \quad - (37)$$

$$\gg f_1(\theta)$$

However the Newtonian result can only be an approximation to the solution of eq. (1). It is the first step in the iteration to the true solution.
