Some basic concepts of fluid dynamics derived from ECE theory

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Basic concepts of flow dynamics are derived from geometry in the context of Einstein Cartan Evans (ECE) theory. These include the concept of the Stokes connection, the continuity equation, and the equations of inviscid and viscous fluids. These concepts can be derived from Cartan's differential geometry using the minimum of hypothesis in accordance with Ockham's Razor.

Keywords: ECE theory, flow dynamics, Stokes connection, continuity equation, inviscid fluid, viscous fluid.

1.   Introduction

The equations of flow dynamics are usually based on the Navier Stokes system of equations, derived independently by Navier and Stokes in the nineteenth century [1]. These are well described in many textbooks. In this paper the Einstein Cartan Evans theory (ECE theory) is applied for the first time to flow dynamics to derive the basic concepts of flow dynamics from Cartan's differential geometry, thus making flow dynamics a theory of general relativity from which its main equations may be derived in well defined limits. In Section 2 the continuity equation of the Navier Stokes system is derived from the tetrad postulate of Cartan's differential geometry [2–11]. It is shown that the Stokes connection is a limiting form of the spin connection of Cartan geometry.

In Section 3 the equation of the inviscid fluid is derived from the first Cartan structure equation, and in Section 4 the analysis is extended to the equation of the viscous fluid. Similarities with electrodynamics are pointed out within the context of ECE theory.

2.   The continuity equation of the Navier–Stokes system

This equation is the equation of conservation of mass density ρ, and is analogous to the equation of continuity of charge density in electrodynamics. The equation

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can be written as [1]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1}$$

where $\mathbf{v}$ is linear velocity. Eq. (1) in tensor notation is:

$$\partial_\mu U^\mu = 0 \tag{2}$$

where:

$$U^\mu = (c\rho, \mathbf{v}\rho) \tag{3}$$

Here c is the vacuum speed of light and the density and velocity are:

$$\rho = \rho \ (X, \ Y, \ Z, \ t) \tag{4}$$

and

$$\mathbf{v} = \mathbf{v} \ (X, Y, Z, t) \tag{5}$$

Equation (1) is:

$$\frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{v} + \rho \nabla \cdot \mathbf{v} = 0 \tag{6}$$

i.e.:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0 \tag{7}$$

where

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{v} \tag{8}$$

is the Stokes derivative [1[. This is a measure of the rate of change of density at a point moving with the fluid. It is the derivative along a path moving with velocity $\mathbf{v}$. In this Section it is shown that the Stokes derivative is a covariant derivative of general relativity and that the continuity of the Navier Stokes

system is a tetrad postulate.
Equation (6) may be written as

$$\frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla + \nabla \cdot \mathbf{v}) \rho = 0 \tag{9}$$

so the covariant derivative is:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla + \nabla \cdot \mathbf{v}) \tag{10}$$

The connection is therefore a simple scalar connection:

$$\Gamma = \mathbf{v} \cdot \nabla + \nabla \cdot \mathbf{v} \tag{11}$$

and Eq. (9) is:

$$\frac{\partial \rho}{\partial t} + \Gamma \rho = 0 \tag{12}$$

which is an equation of general relativity. The Stokes derivative of a velocity $\underline{v}$ is:

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \tag{13}$$

and this concept is used in the other equations of the Navier Stokes system.

   In order to derive Eq. (12) from a unified field theory its basic geometrical structure has to be found. A suitably self consistent geometry is Cartan's differential geometry [2–11], whose well known tetrad postulate is:

$$D_{\mu}\, q_{v}^{a} = \partial_{\mu}\, q_{v}^{a} + \omega_{\mu b}^{a}\, q_{v}^{b} - \Gamma_{\mu v}^{\lambda} q_{\lambda}^{a} = 0 \tag{14}$$

Here q_{v}^{a} is the Cartan tetrad, $\omega_{\mu b}^{a}$ is the spin connection and $\Gamma_{\mu v}^{\lambda}$ is the antisymmetric Riemannian connection:

$$\Gamma_{\mu v}^{\lambda} = q_{a}^{\lambda}\, \Gamma_{\mu v}^{a} = q_{a}^{\lambda} \left(\partial_{\mu} q_{v}^{a} + \omega_{\mu b}^{a} q_{v}^{b} \right) \tag{15}$$

Introduce the mass density tetrad:

$$\rho_{\mu}^{a} = \rho\, q_{\mu}^{a} = \left(\rho_{0}^{a}, \boldsymbol{\rho}^{a} \right) \tag{16}$$

where:

$$a = (0) , (1), (2), (3) \tag{17}$$

is the complex circular basis of spacetime [3–11]. The everyday scalar valued mass density is a limit of this more general concept of mass density tetrad. The vector part of the density tetrad is defined as:

$$\boldsymbol{\rho}^{(1)} = \rho_X^{(1)} \, \mathbf{i} + \rho_Y^{(1)} \, \mathbf{i} + \rho_Z^{(1)} \mathbf{k} \tag{18}$$

and so on. Therefore:

$$\rho_\mu^{(1)} = \left(\rho_0^{(1)}, \, \boldsymbol{\rho}^{(1)} \right) \tag{19}$$

and so on. The timelike part of the mass density tetrad is defined as:

$$\rho_\mu^{(0)} = \left(\rho_0^{(0)}, \mathbf{0} \right) \tag{20}$$

so

$$\boldsymbol{\rho}^{(0)} = \mathbf{0} \tag{21}$$

The complex circular basis is defined in terms of the Cartesian unit vectors i, j, and k as:

$$\mathbf{e}^{(1)} = \frac{1}{\sqrt{2}} (\mathbf{i} - i\,\mathbf{j}), \; \mathbf{e}^{(2)} = \frac{1}{\sqrt{2}} (\mathbf{i} + i\,\mathbf{j}), \; \mathbf{e}^{(3)} = \mathbf{k} \tag{22}$$

so:

$$\boldsymbol{\rho}^{(1)} = \rho_X^{(1)} \, \mathbf{i} + \rho_Y^{(1)} \, \mathbf{i}, \; \boldsymbol{\rho}^{(2)} = \rho_X^{(2)} \mathbf{i} + \rho_Y^{(2)} \, \mathbf{i}, \; \boldsymbol{\rho}^{(3)} = \rho_Z^{(3)} \mathbf{k} \tag{23}$$

are the currents of mass density. Equations (18) to (23) are examples of the fact that the general vector field may always be expressed [12] as:

$$\mathbf{V} = \mathbf{V}^{(1)} + \mathbf{V}^{(2)} + \mathbf{V}^{(3)} \tag{24}$$

an extension of the Helmholtz theorem.

The scalar density may be defined now as the sum of three timelike components of the density tetrad:

$$\rho = \rho_0^{(1)} + \rho_0^{(2)} + \rho_0^{(3)} \tag{25}$$

and the tetrad postulate implies that:

$$D_\mu \rho_\nu^a = 0 \tag{26}$$

Therefore the continuity equation (12) implies that:

$$\frac{\partial}{\partial t}\left(\rho_0^{(1)} + \rho_0^{(2)} + \rho_0^{(3)}\right) + \Gamma\left(\rho_0^{(1)} + \rho_0^{(2)} + \rho_0^{(3)}\right) = 0 \tag{27}$$

It is plausible to assume that:

$$\frac{\partial \rho_0^{(1)}}{\partial t} + \boldsymbol{\Gamma}\rho_0^{(1)} = 0 \tag{28}$$

$$\frac{\partial \rho_0^{(2)}}{\partial t} + \boldsymbol{\Gamma}\rho_0^{(2)} = 0 \tag{29}$$

$$\frac{\partial \rho_0^{(3)}}{\partial t} + \boldsymbol{\Gamma}\rho_0^{(3)} = 0 \tag{30}$$

so the indices in Eq. (26) are determined by:

$$\frac{1}{c}\frac{\partial \rho_0^{(1)}}{\partial t} + \omega_{00}^{(1)} - \boldsymbol{\Gamma}_{00}^{(1)} = 0 \tag{31}$$

$$\frac{1}{c}\frac{\partial \rho_0^{(2)}}{\partial t} + \omega_{00}^{(2)} - \boldsymbol{\Gamma}_{00}^{(2)} = 0 \tag{32}$$

$$\frac{1}{c}\frac{\partial \rho_0^{(3)}}{\partial t} + \omega_{00}^{(3)} - \boldsymbol{\Gamma}_{00}^{(3)} = 0 \tag{33}$$

However:

$$\Gamma^{(1)}_{00} = \Gamma^{(2)}_{00} = \mathbf{\Gamma}^{(3)}_{00} = 0 \tag{34}$$

so:

$$\frac{\partial}{\partial t}\left(\rho^{(1)}_0 + \rho^{(2)}_0 + \rho^{(3)}_0\right) + c\left(\omega^{(1)}_{00} + \omega^{(2)}_{00} + \omega^{(3)}_{00}\right) = 0 \tag{35}$$

i.e.

$$\frac{\partial \rho}{\partial t} + c\left(\omega^{(1)}_{00} + \omega^{(2)}_{00} + \omega^{(3)}_{00}\right) = 0 \tag{36}$$

Comparing Eqs. (12) and (36):

$$\Gamma = c\left(\omega^{(1)}_{00} + \omega^{(2)}_{00} + \omega^{(3)}_{00}\right) \tag{37}$$

The connection of the Stokes derivative is given by the combination (37) of spin connections of Cartan's differential geometry, Q.E.D. so the continuity equation of the Navier Stokes system is a limiting equation of general relativity.

3.   Equation of the inviscid fluid

The inviscid fluid is a well known textbook idealization [1] that nevertheless serves to illustrate how the basics of fluid dynamics may be derived from Cartan's differential geometry. The ECE engineering model [2–11] uses the first Cartan structure equation and one simple hypothesis to produce the following equation for acceleration in dynamics:

$$\mathbf{g}^a = -\frac{\partial \mathbf{v}^a}{\partial t} - c\nabla\ v^a_0 - \omega^a_{0b}\,c\,\mathbf{v}^b + c\mathrm{v}^b_0\,\mathbf{\omega}^a_b \tag{38}$$

and also a general expression for angular velocity:

$$\mathbf{\Omega}^a = \nabla \times \mathbf{v}^a - \mathbf{\omega}^a_b \times \mathbf{v}^b \tag{39}$$

The velocity vector field is expanded as an example of Eq. (24):

$$\mathbf{v} = \mathbf{v}^{(1)} + \mathbf{v}^{(2)} + \mathbf{v}^{(3)} \tag{40}$$

The spin connection in these equations is defined as:

$$\omega_{\mu b}^{a} = \left(\omega_{0b}^{a}, -\underline{\omega}_{b}^{a} \right) \tag{41}$$

and the velocity tetrad as:

$$v_{\mu}^{a} = \frac{1}{c} \Phi_{\mu}^{a} \tag{42}$$

where:

$$\Phi_{\mu}^{a} = \Phi q_{\mu}^{a} \tag{43}$$

is the gravitational potential tetrad.

It will be shown in this Section that Eq. (38) gives the equation of the inviscid fluid as a particular limiting format. In the non relativistic limit [1] acceleration may be defined in a linear approximation of a MacLaurin series as:

$$\mathbf{a} = \left(\nabla \cdot \mathbf{v} \right) \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} \tag{44}$$

and the vorticity [1] defined as:

$$\mathbf{\Omega} = \nabla \times \mathbf{v} \tag{45}$$

The format of Eq. (38) means that acceleration in general relativity is:

$$\mathbf{a}^{a} = \frac{\partial \mathbf{v}^{a}}{\partial t} + \omega_{0b}^{a} \, \mathbf{v}^{b} + c \, \nabla \, v_{0}^{a} - c \, v_{0}^{b} \mathbf{\omega}_{b}^{a} \tag{46}$$

and that vorticity in general relativity is:

$$\mathbf{\Omega}^{a} = \nabla \times \mathbf{v}^{a} - \mathbf{\omega}_{b}^{a} \times \mathbf{v}^{b} \tag{47}$$

In the equations the four velocity is:

$$v_{\mu}^{a} = \left(v_{0}^{a}, -\mathbf{v}^{a} \right) = \frac{1}{c} \left(\Phi_{0}^{a}, -\mathbf{\Phi} \right) \tag{48}$$

In an inviscid fluid [1] the acceleration is calculated as:

$$\mathbf{a} = -\frac{1}{m}\left(\nabla p + \nabla \phi\right) \tag{49}$$

where p is the pressure, m is the mass, and 0 is the total potential energy per unit mass due to all body forces. So the textbook description of an inviscid fluid is [1]:

$$\frac{\partial \mathbf{v}}{\partial t} + \left(\nabla \cdot \mathbf{v}\right)\mathbf{v} = -\frac{1}{m}\left(\nabla p + \nabla \phi\right) \tag{50}$$

or:

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \left(\nabla \cdot \mathbf{v}\right)\mathbf{v} + \frac{1}{m}\left(\nabla p + \nabla \phi\right) = 0 \tag{51}$$

and the net acceleration is zero.

In general relativity for zero net acceleration Eq. (46) is:

$$\frac{\partial \mathbf{v}^a}{\partial t} + \omega^a_{0b}\,\mathbf{v}^b + c\nabla \mathbf{v}^a_0 - c\mathbf{v}^b_0 \omega^a_b = \mathbf{0} \tag{52}$$

This equation may be simplified to:

$$\frac{\partial \mathbf{v}^a}{\partial t} + \mathbf{v}\mathbf{\Omega}^a + c\nabla \mathbf{v}^a_0 - c\,\Phi\,\mathbf{\omega}^a = 0 \tag{53}$$

where:

$$\mathbf{\omega}^a = \omega^a_{10}\,\mathbf{i} + \omega^a_{20}\,\mathbf{j} + \omega^a_{30}\,\mathbf{k} \tag{54}$$

and

$$\mathbf{\Omega}^a = \omega^a_{01}\,\mathbf{i} + \omega^a_{02}\,\mathbf{j} + \omega^a_{03}\,\mathbf{k} \tag{55}$$

In this simplification we have used:

$$\mathbf{v}^b_0 \mathbf{\omega}^a_b = \mathbf{v}\,\mathbf{\omega}^a \tag{56}$$

and

$$\omega_{0b}^a \, \mathbf{v}^b = v \boldsymbol{\Omega}^a \tag{57}$$

For each a in Eq. (53)

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + v \boldsymbol{\Omega} + c \nabla v_0 - c \, v \boldsymbol{\omega} \tag{58}$$

and this reduces to Eq. (51) if:

$$v \underline{\Omega} = \left(\underline{\nabla} \cdot \underline{v} \right) \underline{v} \tag{59}$$

$$\frac{1}{m} \left(\nabla \rho + \nabla \phi \right) = c \left(\nabla v_0 - v \boldsymbol{\omega} \right) \tag{60}$$

So we have derived the textbook inviscid fluid equation from the Cartan structure equation with the minimum use of hypothesis, Q.E.D.

Note carefully that this procedure uses the spin connection as an intrinsic part of the fluid dynamics, so general relativity itself is extended considerably as a subject. It is no longer a small correction to Newtonian dynamics but an intrinsic part of fluid dynamics. All equations of fluid dynamics become equations of general relativity and some equations of fluid dynamics may be corrected using the concept of spin connection. One of these is the vorticity equation, which in general relativity is:

$$\boldsymbol{\Omega}^a = \nabla \times \mathbf{v}^a - \boldsymbol{\omega}_b^a \times \mathbf{v}^b \tag{61}$$

but which in the textbooks is [1]:

$$\boldsymbol{\Omega} = \nabla \times \mathbf{v} \tag{62}$$

Eq. (60) in the philosophy of general relativity is a correction of Eq. (61). Accepting the textbook eq. (61) for the sake of illustration only, the usual procedure [1] is to use Eqs. (51) and (61) to produce the textbook equation of motion of the inviscid fluid:

$$\frac{\partial \underline{\Omega}}{\partial t} + \nabla \times \left(\boldsymbol{\Omega} \times \mathbf{v} \right) = 0 \tag{63}$$

using the assumption:

$$\nabla \cdot \mathbf{v} = 0 \tag{64}$$

General relativity shows that there are terms missing in Eq. (63). These terms may lead to effects which are experimentally observable.
Accepting Eq. (63) again just for the sake of illustration, it has the same structure as the textbook Faraday law of induction:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{65}$$

where $\mathbf{B}$ is the magnetic flux density and $\mathbf{E}$ is the electric field strength. Equation (62) has the structure:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{66}$$

of electrodynamics in the standard model. In ECE theory Eq. (66) becomes:

$$\mathbf{B}^a = \nabla \times \mathbf{A}^a - \boldsymbol{\omega}_b^a \times \mathbf{A}^b \tag{67}$$

Eq. (64) is analogous to:

$$\nabla \cdot \mathbf{A} = 0 \tag{68}$$

in the standard model of electrodynamics. Using the minimal prescription:

$$\rho = m\mathbf{v} = e\mathbf{A} \tag{69}$$

where m is mass and e is charge, the standard model of electrodynamics gives:

$$\mathbf{E} = \frac{e}{m}\mathbf{B} \times \mathbf{A} = \mathbf{B} \times \mathbf{v} \tag{70}$$

which is the Lorentz force law. In summary:

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + \nabla \times (\boldsymbol{\Omega} \times \mathbf{v}) = \mathbf{0} \quad \Leftrightarrow \quad \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0}$$

$$\boldsymbol{\Omega} = \nabla \times \mathbf{v} \quad \Leftrightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \cdot \mathbf{v} = 0 \quad \Leftrightarrow \quad \nabla \cdot \mathbf{A} = 0$$

The quantity $\mathbf{\Omega} \times \mathbf{v}$ is analogous to the Coriolis acceleration and to the electric field strength $\mathbf{E}$. The vorticity $\mathbf{\Omega}$ plays the role of the magnetic flux density $\mathbf{B}$. The velocity $\mathbf{v}$ plays the role of the vector potential $\mathbf{A}$.

Note carefully that these concepts are extended and self consistently generalized in ECE theory in which:

$$\mathbf{a}^a = \frac{\partial \mathbf{v}^a}{\partial t} + c\,\omega_{0b}^a\,\mathbf{v}^b + c\,\nabla \mathbf{v}_0^a - c\,\mathbf{v}_0^b\,\omega_b^a \tag{71}$$

$$\mathbf{\Omega}^a = \nabla \times \mathbf{v}^a - \omega_b^a \times \mathbf{v}^b \tag{72}$$

The expression for acceleration used in Eq. (44) is:

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t}(\nabla \cdot \mathbf{v})\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} \tag{73}$$

and is a special case of:

$$\mathbf{a}_{\mathrm{I}}^a = \frac{\partial \mathbf{v}^a}{\partial t} + c\,\omega_{0b}^a\,\mathbf{v}^b \tag{74}$$

Eq. (49) is a special case of:

$$\mathbf{a}_{\mathrm{II}}^a = c\left(\nabla\,\mathbf{v}_0^a - \mathbf{v}_0^b\omega_b^a\right) \tag{75}$$

and the textbook inviscid, incompressible fluid is:

$$\mathbf{a}_{\mathrm{I}}^a + \mathbf{a}_{\mathrm{II}}^a = \mathbf{0} \tag{76}$$

General relativity is no longer a minor correction to Newtonian dynamics, but becomes in ECE theory an intrinsic part of everyday dynamics, adding new terms to well known equations, terms which may have a measurable experimental effect or which are given by well known effects interpreted in a new way.

4. Viscous fluid equation

In this case there is a viscous force $\mathbf{f}_v$, so Eq. (51) becomes [1]:

$$\mathbf{f}_v = \rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right) + \nabla p + \rho\nabla\phi \tag{77}$$

The most general form of second derivatives that can occur in a vector equation is a linear combination of terms $\nabla^2\mathbf{v}$ and $\nabla(\nabla\cdot\mathbf{v})$, so the viscous force is expressed as:

$$\mathbf{f}_v = \mu\nabla^2\mathbf{v} + (\mu+\mu')\nabla(\nabla\cdot\mathbf{v}) \tag{78}$$

where μ and μ' are coefficients. So the viscous fluid equation of motion is derived using the vorticity (62) and is:

$$\frac{\partial\boldsymbol{\Omega}}{\partial t} + \nabla\times(\boldsymbol{\Omega}\times\mathbf{v}) = \frac{\mu}{\rho}\nabla^2\boldsymbol{\Omega} \tag{79}$$

In ECE theory and general relativity there are again new terms for the viscous fluid and for other equations of the Navier Stokes system. These will be developed in later papers.

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