DERIVATION OF A LOCALLY GAUGE INVARIANT PROCA
EQUATION FROM U(1) AND O(3) GAUGE THEORY APPLIED TO
VACUUM ELECTRODYNAMICS; ACQUISITION OF PHOTON
MASS AND REST ENERGY FROM THE VACUUM.

by

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ABSTRACT

Electrodynamics in the vacuum is considered as a U(1) and O(3) invariant gauge theory. In both cases local gauge transformation results in a vacuum charge / current density. A Higgs mechanism is used to derive a locally gauge invariant Proca equation in a U(1) and O(3) invariant electrodynamics. Therefore the photon acquired mass and rest energy from the vacuum as it propagates. The advantages of an O(3) over a U(1) invariant gauge theory applied to vacuum electrodynamics are discussed.

KEYWORDS: Locally gauge invariant Proca equation, vacuum charge / current density; photon mass; Higgs mechanism.
1. INTRODUCTION

Gauge theory applied to electrodynamics in the vacuum requires the presence of an internal gauge space \(1-3\). In a \(U(1)\) invariant gauge theory of any type the internal space is characterized by a complex field with two components in a plane and in an \(O(3)\) invariant theory by a complex field with three components in three-dimensional space. The \(\omega\) of a complex field indicates that the particle concomitant with the field is charged. These concepts may be applied to electromagnetism in the vacuum by considering a topological charge, \(g\), which appears in the covariant derivative obtained by local gauge transformation and by considering the components of the field in the internal space to be components of the vector potential in the vacuum.

From this starting point a globally invariant lagrangian is constructed in Section 2 both on the \(U(1)\) and \(O(3)\) invariant levels. From this lagrangian, the wave equation in the internal space of the gauge theory is obtained in the vacuum. In Section 3 the locally invariant lagrangian is obtained from a local gauge transformation and Euler-Lagrange equations used to derive the locally gauge invariant wave and field equations. The latter contain a vacuum charge current density both in a \(U(1)\) and \(O(3)\)
2. GLOBALLY INVARIANT LAGRANGIANS AND WAVE EQUATIONS IN THE INTERNAL SPACES OF THE U(1) AND O(3) INVARIANT GAUGE THEORIES.

The theory is first developed in U(1) invariant form \( \{ 1 - 3 \} \) by considering a complex field made up of two components of the vector potential:

\[
\begin{align*}
\mathbf{A} &= \frac{1}{\sqrt{2}} \left( \mathbf{A}_1 + i \mathbf{A}_2 \right) \quad (1) \\
\mathbf{A}^\ast &= \frac{1}{\sqrt{2}} \left( \mathbf{A}_1 - i \mathbf{A}_2 \right) \quad (2)
\end{align*}
\]

The complex field \( \mathbf{A} \) and its conjugate \( \mathbf{A}^\ast \) are considered to be independent fields, and signal the existence of a topological charge \( \{ 7 - 1a \} \):

\[
\mathcal{Q} = \frac{\kappa}{\mathbf{A}^{(\infty)}} \quad (3)
\]
where \( \chi \) is the wave-number and \( A \) a vector potential magnitude. The topological charge defines the U(1) invariant covariant derivative:

\[
\mathcal{J}_\mu = \partial_\mu + i \mathcal{A}_\mu \quad - (2)
\]

after local gauge transformation \((1 - 3)\). The complex fields \((1)\) and \((2)\) define the globally invariant lagrangian:

\[
\mathcal{L} = \left( \frac{\partial A}{\partial \xi} \right) \left( \partial^\ast A^\ast \right) \quad - (4)
\]

and the Euler Lagrange equations:

\[
\frac{\partial \mathcal{L}}{\partial A} = \frac{\partial}{\partial \xi} \left( \frac{\partial \mathcal{L}}{\partial (\partial A)} \right); \quad \frac{\partial \mathcal{L}}{\partial A^\ast} = \frac{\partial}{\partial \xi} \left( \frac{\partial \mathcal{L}}{\partial (\partial A^\ast)} \right) \quad - (5)
\]

produce d'Alembert wave equations for each component:

\[
\Box A = \Box A^\ast = 0. \quad - (6)
\]

Local gauge transformation is now applied, defined by:

\[
A \rightarrow \exp \left( - i x^\ast (A^\ast) \right) A; \quad A^\ast \rightarrow \exp \left( i x A^\ast \right) A^\ast \quad - (7)
\]

to give the locally gauge invariant lagrangian \((1 - 3)\)

\[
\mathcal{L} = \mathcal{D}_\mu A \mathcal{B}^\mu A^\ast - \frac{1}{4} F_{\mu \nu}^{\ast} F^{\mu \nu} \quad - (8)
\]
from which the Euler Lagrange equations (5) give the locally gauge invariant wave equations in the vacuum:

\[ D^\nu (D_\nu A^\mu) = 0 \quad - (9) \]

\[ D_\nu (\partial^\nu A^\mu) = 0 \quad - (10) \]

where \( A^\mu \) is the four potential and where

\[ F_{\mu\nu} = \partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu} \quad - (11) \]

is the electromagnetic field in the vacuum.

The Euler Lagrange equation:

\[ D_\nu \left( \frac{\delta S}{\delta A^\mu} \right) \sim \frac{\delta S}{\delta A^\mu} \quad - (12) \]

gives the Lehnert equation \((12)\) in S.I. units:

\[ D_\nu F_{\mu\nu} = -i \frac{\gamma}{c} \left( A^* D^\mu A - A D^\mu A^* \right) \quad - (13) \]

where \( D_\nu \) is the covariant derivative defined by \((1-3)\):

\[ D_\nu A = (\partial_\nu + ig A_\nu) A \quad - (13) \]

\[ D_\nu A^* = (\partial_\nu - ig A_\nu) A^* \quad - (15) \]
Therefore if $U(1)$ invariant gauge theory is applied rigorously to electromagnetism in the vacuum there appears a vacuum charge current density:

$$\mathcal{J}^\mu (\nu \alpha \omega) = -i \varepsilon_{\alpha \beta \gamma} \left( A^\mu \mathcal{D}_\nu A - A^\mu \mathcal{D}_\nu A \right)$$

and the wave equations become (\ref{eq:11}) and (\ref{eq:12}). A covariant derivative appears containing the topological charge (\ref{eq:9}).

The basis of this development is that in a $U(1)$ invariant gauge there must be an internal space of this symmetry. The only field present is the electromagnetic field, so the internal space must define the complex scalar fields (\ref{eq:14}) and (\ref{eq:22}).

In the usual Maxwell-Heaviside theory, the Lehner charge current density (\ref{eq:16}) is missing, and there is no topological charge present in the theory, so covariant derivatives are replaced by ordinary derivatives and the vacuum d’Alembert equations (\ref{eq:11}) and (\ref{eq:12}) become:

$$\Box A^\mu = 0$$
$$\Box A = 0$$
$$\Box A^b = 0$$
There is no internal gauge space present in the Maxwell-Heaviside theory and therefore the theory cannot be described self-consistently as $U(1)$ invariant in the vacuum. The structure of the wave equations (7.10) and (7.11) do not correspond to that of a Proca equation.

In an $O(3)$ invariant vacuum electrodynamics (7.12) the globally invariant Lagrangian is:

$$
\mathcal{L} = \gamma \cdot A \cdot A^* - (\ddot{A})^2
$$

where $A$ and $A^*$ are vectors in the $O(3)$ symmetry internal gauge space of the theory. These are independent complex vectors and the concomitant charge is again the topological charge defined in eqn. (3). The Euler Lagrange equations:

$$
\frac{\partial \mathcal{L}}{\partial \dot{A}} = \partial_{\tau} \left( \frac{\partial \mathcal{L}}{\partial \ddot{A}} \right) ; \quad \frac{\partial \mathcal{L}}{\partial \dot{A}^*} = \partial_{\tau} \left( \frac{\partial \mathcal{L}}{\partial \ddot{A}^*} \right)
$$

give the globally invariant d'Alambert wave equations:

$$
\Box A = 0 ; \quad \Box A^* = 0 .
$$

A local gauge transformation on the $O(3)$ level is defined by (7.13):
where \( J_i \) are notation generators of the O(3) group and where \( A_i \) are angles. The gauge transformation (3.3) gives the locally gauge invariant lagrangian:

\[
\mathcal{L} = e^{-i J_i A_i^*} \mathcal{L}; \quad A_i^* \rightarrow e^{-i J_i A_i^*} \mathcal{L} = \mathcal{L}(2.3)
\]

The Euler Lagrange equation:

\[
\frac{\partial \mathcal{L}}{\partial A_i} - \frac{\partial}{\partial x^i} \left( \frac{\partial \mathcal{L}}{\partial (\partial A_i^* / \partial x^i)} \right) = 0
\]

gives the O(3) invariant inhomogeneous field equation in the vacuum:

\[
\frac{\partial}{\partial x^i} A_i^{\mu \nu} = -\gamma \frac{\partial}{\partial x^i} A_i^{\mu \nu} \times A_i^* - \mathcal{L}(2.4)
\]

where the right hand side defines the O(3) invariant Lehnert charge current density.

The lagrangian (2.14) can be developed as:

\[
\mathcal{L} = \frac{1}{2} \partial \mu A_i^* \cdot \partial \nu A_i^* + \cdots
\]

\[
= \left( \partial \mu + \partial \nu \times A_i^* \right) A_i^* \cdot \left( \partial \nu - \partial \mu \times A_i^* \right) A_i^* + \cdots
\]

\[
= \frac{1}{2} \partial \mu A_i^* \cdot \partial \nu A_i^* + \frac{1}{2} \partial \mu \times A_i^* \cdot \partial \nu A_i^* + \frac{1}{2} \partial \mu \times A_i^* \cdot \partial \nu A_i^* + \gamma \left( \partial \mu \times A_i^* \right) \cdot \left( \partial \nu \times A_i^* \right)
\]

\[
+ \cdots
\]
gives the O(3) invariant wave equations:

\[ \begin{align*}
(\partial^2 - \Lambda \kappa^2) (\partial \kappa + \Lambda \partial \kappa x) \kappa &= 0 \\
\partial \kappa + \Lambda \partial \kappa x (\partial \kappa - \Lambda \kappa^2 \kappa) &= 0
\end{align*} \quad \text{-(28)} \]

These are different in structure from their U(1) counterparts, eqns. \text{(9)} and \text{(10)}. In condensed notation \text{(1-3)}, eqns. \text{(28)} and \text{(29)} can be written as:

\[ \begin{align*}
(\partial \kappa - \Lambda \kappa^2 \kappa) (\partial \kappa + \Lambda \partial \kappa x) &= 0 \\
(\partial \kappa + \Lambda \partial \kappa x) (\partial \kappa - \Lambda \kappa^2 \kappa) &= 0
\end{align*} \quad \text{-(30)} \]

and developed as:

\[ \begin{align*}
(\partial \kappa - \Lambda \kappa^2 \kappa)^2 - \Lambda^2 \partial \kappa^2 \kappa^2 \kappa + \Lambda^2 \partial \kappa^2 \kappa^2 \kappa &= 0 \\
(\partial \kappa + \Lambda \partial \kappa x)^2 + \Lambda^2 \partial \kappa^2 \kappa^2 \kappa &= 0
\end{align*} \quad \text{-(31)} \]

Using the quantum ansatz

\[ p_{\mu} = i \frac{\hbar}{\kappa} \partial_{\mu} \]

eqns. \text{(3,2)} and \text{(3,3)} simplify to:
which have the form of Proca equations. Using eqn. (3) the O(3) invariant Proca equations are:

\[
\begin{align*}
\left( \Box + \gamma^2 \right) A^\mu \dot{A}^\mu &= 0 \quad &\text{(35)} \\
\left( \Box + \gamma^2 \right) \dot{A} &= 0 \quad &\text{(36)}
\end{align*}
\]

Finally, the de Broglie Guidance Theorem:

\[
\frac{\nabla \cdot \dot{r}}{c^2} = m_0 c^2 \quad \text{(39)}
\]

gives the O(3) Proca equations in the form:

\[
\begin{align*}
\left( \Box + \frac{m_0 c^2}{\xi^2} \right) A^\mu \dot{A}^\mu &= 0 \quad &\text{(40)} \\
\left( \Box + \frac{m_0 c^2}{\xi^2} \right) \dot{A} &= 0 \quad &\text{(41)}
\end{align*}
\]

where \( m_0 \) is the rest mass of the photon.
It is shown in this section that the introduction of a Higgs mechanism (spontaneous symmetry breaking of the vacuum) produces vacuum charge current densities in addition to the Lehner type, which as we have seen in section (2), is produced by a local gauge transformation.

In a U(1) invariant theory the Higgs mechanism is introduced through the globally invariant lagrangian:

$$\mathcal{L} = \mathcal{T} - \mathcal{V} = (\partial \mu A^\mu)(\partial \nu A^\nu) - m^2 A^\mu A^\mu - \frac{1}{2} \lambda (A^\mu A^\mu)^2$$

from which is obtained

$$\frac{\partial \mathcal{V}}{\partial A} = m^2 A^\mu + \frac{\lambda}{2} A^\mu (A^\nu A^\nu).$$

If $m^2 < 0$ there is a local maximum at $A^\mu = 0$ and a minimum at

$$A^\mu = \frac{1}{\sqrt{2 \lambda}}. $$

The scalar fields $\mathcal{A}$ and $A^\mu$ therefore become:

$$\mathcal{A}(x^\nu) = A^\mu + \frac{1}{\sqrt{2 \lambda}} (A^\mu + i A^\mu)$$

and the lagrangian becomes:

$$\mathcal{L} = \frac{1}{2} \partial \mu (\mathcal{A} + A^\mu) \partial \nu (\mathcal{A} + A^\mu) - m^2 (A + A^\mu)(A + A^\mu)$$

$$- \frac{\lambda}{2} ((A + A^\mu)(A + A^\mu))^2.$$
which can be developed as:

$$\mathcal{L} = \lambda \rho^2 A^2 - 2 \lambda a^2 A_3^2 - \sqrt{2} \lambda a_3 (A_2^2 + A_3^2) - \frac{\lambda}{4} (A_2^2 + A_3^2)^2 - 2 \lambda a_4.$$

The Higgs mechanism has therefore acted in such a way as to produce a globally invariant field component $A_1$ with mass.

A local gauge transformation of the lagrangian (14.7) produces the locally invariant lagrangian:

$$\mathcal{L} = (\mathcal{L}_m + i \alpha A^\mu)(a + \mathcal{L}_m)(\mathcal{L}_m - i \alpha A^\mu)(a + \mathcal{L}_m)$$

$$- \frac{1}{2} (a + \mathcal{L}_m)^2 - \lambda (a + \mathcal{L}_m)^2 (a + \mathcal{L}_m)^2$$

$$- \frac{1}{4} F^{\mu\nu} F_{\mu\nu}.$$  

which when used in eqn. (12) produces the field equation:

$$\partial_\mu F^{\mu\nu} = -i \alpha (A^\mu \partial_\mu a - A a^\mu \partial_\mu) - \frac{2}{\lambda} a^\mu A^\mu + \sqrt{2} \alpha \delta^\mu_2 a_3 A_2.$$

The term $-\frac{2}{\lambda} a^\mu A^\mu$ implies that the electromagnetic four potential has acquired mass in a U(1) invariant gauge theory. All four vacuum charge current densities produce vacuum energy through the equation:

$$\mathcal{E}_0 (\text{vac}) = \int \mathcal{F}_\mu (\text{vac}) A_\mu dV.$$
The locally invariant lagrangian can be written out as

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\alpha}{2} A^\mu A_\mu + \frac{1}{2} \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right)^2 + \frac{1}{2} \left( \partial_\mu A_\nu \right)^2 \]

and at its minimum value simplifies to:

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\alpha}{2} A^\mu A_\mu \]

If the mass of the photon is defined by:

\[ m_\gamma^2 = \frac{\alpha}{2} A^\mu A_\mu \]

then the lagrangian (53) is a U(1) and locally invariant lagrangian for the Proca equation:

\[ \partial_\mu F^{\mu\nu} + m_\gamma^2 A^\nu = 0 \]

Therefore the photon in this U(1) invariant theory has picked up mass from the vacuum using a Higgs mechanism.
In an O(3) invariant theory the starting point is the globally invariant lagrangian:

\[ L = \frac{1}{2} \partial_{\mu} A^a \cdot \partial^{\mu} A^a - \frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} - m^2 A^a \cdot A^a - \lambda (A^a \cdot A^a)^2 \]  

(S6)

from which we obtain:

\[ \frac{\partial L}{\partial A^a} = -m^2 A^a + 2 \lambda A^a (A^a \cdot A^a) \]  

(S7)

\[ \frac{\partial L}{\partial \partial A^a} = -m^2 \partial A^a - 2 \lambda \partial (A^a \cdot A^a) \]  

(S8)

from the Euler Lagrange equations (S1) and (S2). At the Higgs minimum (the symmetry broken vacuum):

\[ A^a \cdot A^a = - \frac{m^2}{2 \lambda} \equiv \alpha^2 \]  

(S9)

and the wave equations (S7) and (S8) reduce to:

\[ \Box A^a = 0 \]  

(S10)

The locally invariant lagrangian obtained from eqn. (S6) is:

\[ L = \frac{1}{2} \partial_{\mu} A^a \cdot \partial^{\mu} A^a - \frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} - m^2 A^a \cdot A^a - \lambda (A^a \cdot A^a)^2 \]  

(S11)
The Euler Lagrange equation \( \frac{\partial \mathcal{L}}{\partial \dot{a}^\mu} = -\frac{\partial S}{\partial a^\mu} \) gives the field equation:

\[
\partial \mu \left( \frac{\partial}{\partial x^\mu} a^\nu - \frac{1}{2} g a^\nu \times a^\mu \right) = - \frac{\partial S}{\partial a^\mu}
\]

with the Lehner charge current density on the right hand side. At the Higgs minimum this charge current density is obtained from the symmetry broken vacuum and takes the form:

\[
\partial \mu \left( \frac{\partial}{\partial x^\mu} a^\nu - \frac{1}{2} g a^\nu \times a^\mu \right) = - \frac{\partial S}{\partial a^\mu}
\]

which is an O(3) invariant Proca equation corresponding to the lagrangian:

\[
\mathcal{L} = -\frac{1}{4} \left( \partial a^\mu \right) \cdot \left( \partial a^\mu - \frac{1}{2} g (a^\mu \times a^\nu) \cdot (a^\mu \times a^\nu) \right)
\]

The mass of the photon in the O(3) invariant theory is derived from the Higgs vacuum, which is the minimum of the potential energy used in the lagrangian \( \mathcal{S} \). The field equation \( \mathcal{L}_4 \) and lagrangian \( \mathcal{L}_5 \) are O(3) invariant and so the existence of photon mass becomes compatible
with the existence of the Evans Vigier field $B = (7 - 1 \mathbf{A})$ and the $O(3)$
invariant B Cyclic theorem. The Higgs mechanism is the basis of much of contemporary elementary particle theory, and this derivation is based on a rigorously $O(3)$ and locally invariant gauge theory.

DISCUSSION

It has been shown in this paper that an $O(3)$ invariant Proca equation can be obtained from a local $O(3)$ invariant gauge transformation (eqn. (20) to (41)). In this mechanism the photon mass appears from the $O(3)$ invariant local gauge transformation itself.

This does not occur in a $U(1)$ invariant gauge theory applied to electrodynamics. It has also been shown that a Higgs mechanism applied in both a $U(1)$ and $O(3)$ invariant theory produces a gauge invariant Proca equation. In this mechanism the photon mass is picked up from the symmetry broken vacuum defined by the Higgs minimum.

The received view of the Proca equation $(1 - 3)$ is that it is not invariant under local gauge transformation, but in this paper it has been shown that the received view is incorrect. Photon mass is
compatible with rigorous U(1) and O(3) invariant gauge theory applied to
electrodynamics in the vacuum. The same procedures produce self-
consistently the Lehnrert charge current densities in the vacuum, and its
concomitant inherent energy, defined by eqn. (51). Therefore there is
energy inherent in the vacuum both in a U(1) and O(3) invariant gauge
theory.

The rigorous application of gauge theory to any problem requires
an internal gauge space, and this is also true of gauge theory applied to
electrodynamics in the vacuum. The internal gauge space has been
defined in this paper through complex scalar (U(1)) and vector (O(3))
fields with a concomitant topological charge (7-13) defined by eqn. (3)
in both cases. If the existence of this internal space is neglected, the
Lehrnetr vacuum charge current density disappears.

It is now known that the advantages of using an O(3) invariant
gauge theory applied to vacuum electrodynamics ("O(3)
electrodynamics") are overwhelming (7-12). For example
interferometric effects are described with precision, one prominent
example being the Sagnac effect (13), another example being
Michelson interferometry (14). A U(1) invariant gauge theory applied
to vacuum electrodynamics ("U(1) electrodynamics") fails to describe
either effect (13, 14), and fails to describe physical optics in general.
O(3) electrodynamics is homomorphic with Barrett's SU(2)
electrodynamics (15), and both have been tested extensively against
empirical data (7-15). The phase factor in both O(3) and SU(2)
electrodynamics is a Wu Yang phase factor, which is related to the Evans
Vigier field (16) (17-19a) using a non-Abelian Stokes Theorem. It has
been shown (19b-19c) that all interferometric effects are topological in
nature, and defined through the topological charge (3). In contrast
U(1) electrodynamics fails to describe the Sagnac effect because its phase
is invariant under motion reversal symmetry, which generates the
anticlockwise from the clockwise loop in the Sagnac effect with platform
at rest (13). U(1) electrodynamics fails to describe Michelson
interferometry because its phase factor is invariant under parity inversion
symmetry, which is equivalent to normal reflection (12). O(3)
electrodynamics explains Michelson interferometry and reflection through
a Wu Yang phase factor (12), and is also successful in explaining
topological effects such as the rotation of the plane of linearly polarized
light propagating through a helix (16). U(1) electrodynamics has no
explanation for this effect. The Aharonov-Bohm effect can be explained by $O(3)$ electrodynamics $\{1\mathbb{A}, 1\mathbb{T}\}$, whereas $U(1)$ electrodynamics fails to give a satisfactory effect. The above mentioned is a selection of many effects $\{\mathbb{I} - 1\mathbb{T}\}$ which $O(3)$ explains but $U(1)$ does not.
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